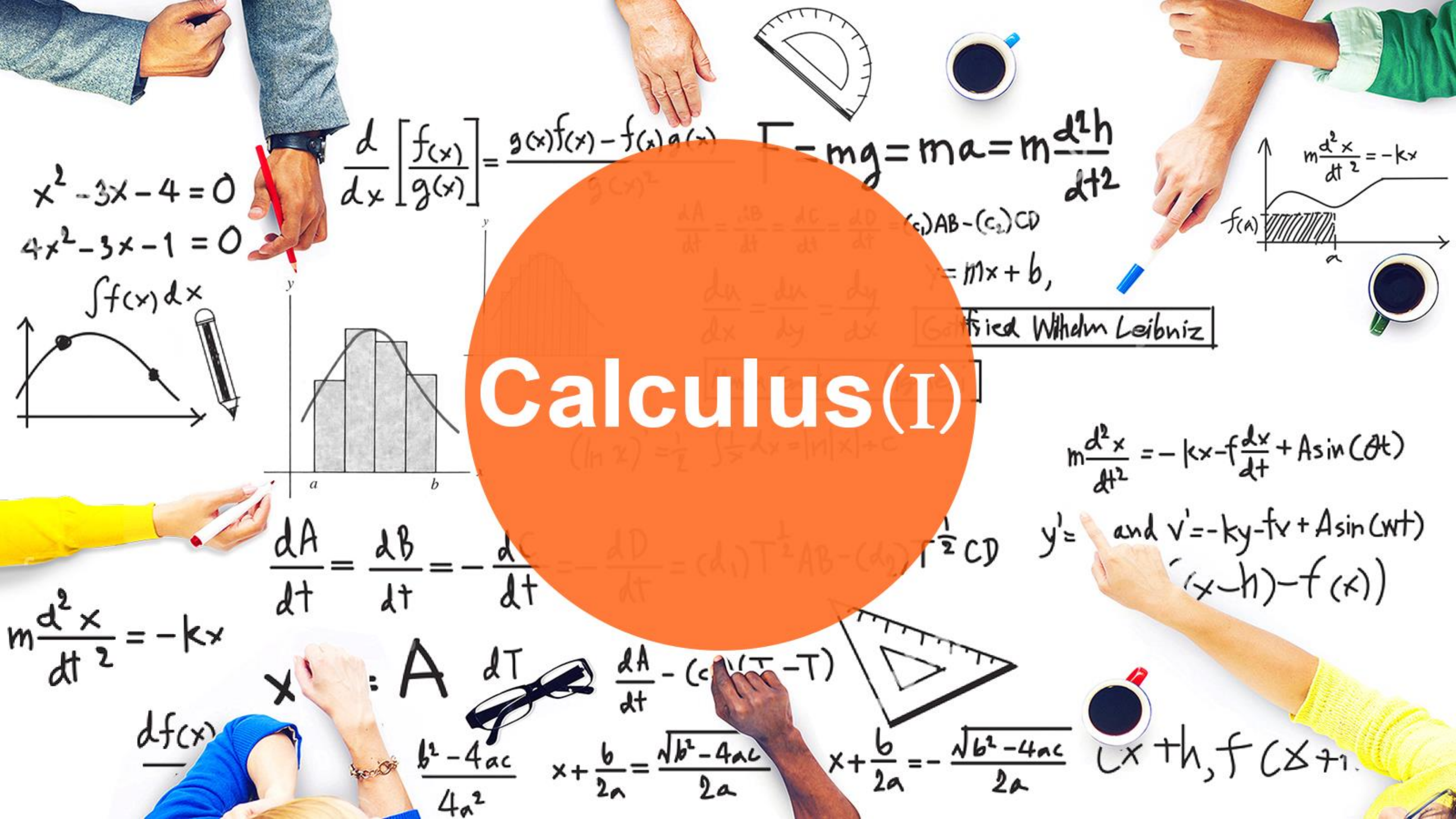
Calculus(Ⅰ)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x)dx
d/dx [f(x)/g(x)] =
F = mg = ma = m d²h/dt²
m d²x/dt² = -kx
f(a)
a
y = mx + b,
Gottfried Wilhelm Leibniz
m d²x/dt² = -kx - f dx/dt + Asin(θt)
y' = and v' = -ky - fv + Asin(wt)
(x-h) - f(x))
dA/dt = dB/dt = - dC/dt
CD
m d²x/dt² = -kx
dT
dA/dt - (c
df(x)
b² - 4ac / 4a²
x + b/2a = √b² - 4ac / 2a
x + b/2a = - √b² - 4ac / 2a

Rigorous Study of Limits

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Symbol Description

1. ε (epsilon) Greek letter stands for (usually small) arbitrary positive number
2. δ (delta) Greek letter stands for (usually small) arbitrary positive number
3. $\forall$ For any
4. $\exists$ Exist, there exists at least one
5. $|f(x)-L|<\varepsilon$ $f(x)$ is within ε of L, $f(x)$ lies in the open interval $(L-\varepsilon, L+\varepsilon)$
6. $|x-c|<\delta$ The interval $c-\delta<x<c+\delta$ including c (x is sufficiently close to c)
7. $0<|x-c|<\delta$ Requires that $x=c$ be excluded (x is sufficiently close to but different from c)

Symbol Description

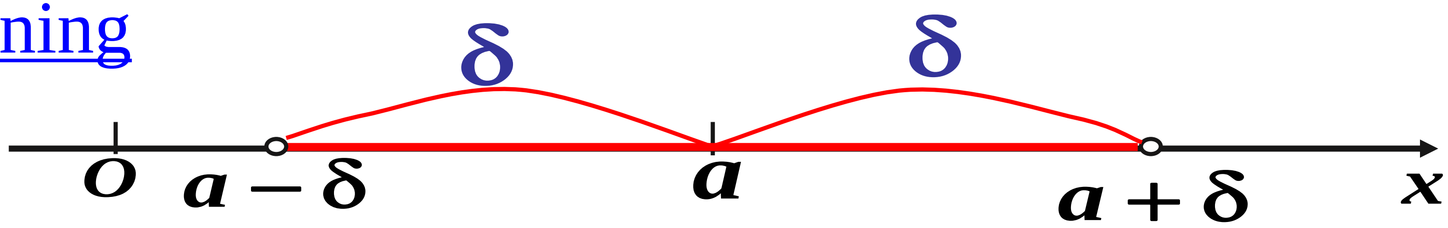
“neighborhood” Let a and δ be real numbers and $\delta > 0$.

Set $\{x \mid |x - a| < \delta\}$ be the δ-neighborhood of point a, which is denoted as

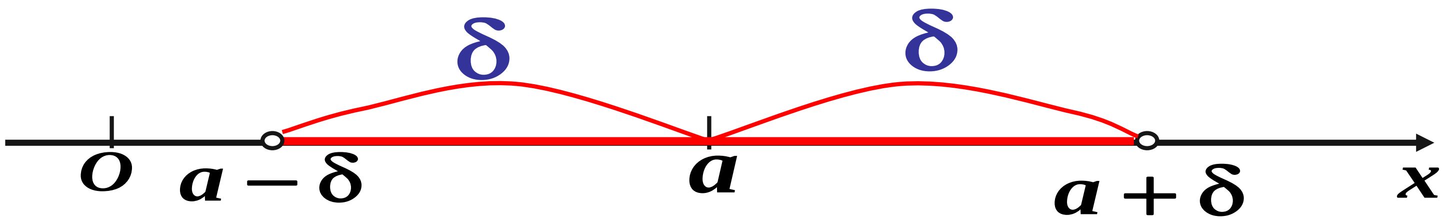
$$U(a, \delta) = \{x \mid a - \delta < x < a + \delta\}.$$

a is regarded as center; δ is regarded as radius

Geometrical meaning



Symbol Description



1. $U(a,\delta)=\{x \mid a-\delta<x<a+\delta\}.$

   Sometimes $U(a,\delta)$ is written as $\boldsymbol{U(a)}$.

2. $\mathring{U}(a,\delta)=\{x \mid 0<|x-a|<\delta\}.$ ($0<|x-a|$: $x \neq a$)

3. the open interval $(a-\delta,a)$ the left neighborhood of δ

   the open interval $(a,a+\delta)$ the right neighborhood of δ

Problem Introduction

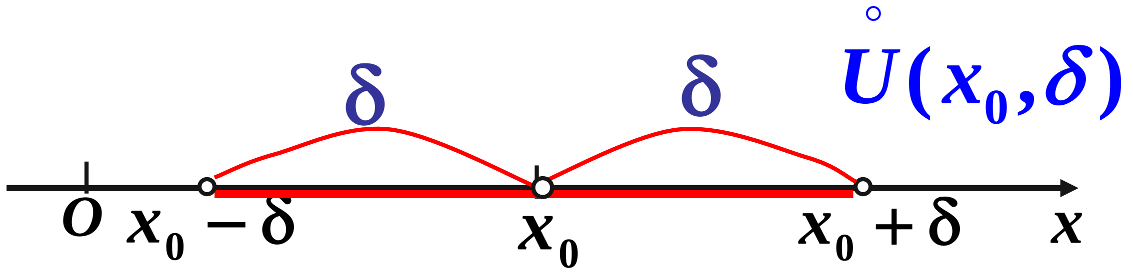
In mathematical language, how to express:

$$x \to x_0, \quad f(x) \to L$$

$\forall \varepsilon > 0, \quad |f(x) - L| < \varepsilon \quad f(x)$ is as close as we like to L;

$\exists \delta > 0, \quad 0 < |x - x_0| < \delta \quad x$ is sufficiently close to but different from x_0.

$x \neq x_0$



Definition Precise Meaning of Limit $(\varepsilon-\delta)$

If $\forall \varepsilon > 0$, $\exists \delta > 0$, such that when $0 < |x - x_0| < \delta$, we have

$$|f(x) - L| < \varepsilon$$

Namely, $\lim_{x \to x_0} f(x) = L$, or $f(x) \to L \ (x \to x_0)$.

That is, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\varepsilon > 0$: no matter how small, there is a correspongding δ>0.

Definition Precise Meaning of Limit $(\varepsilon-\delta)$

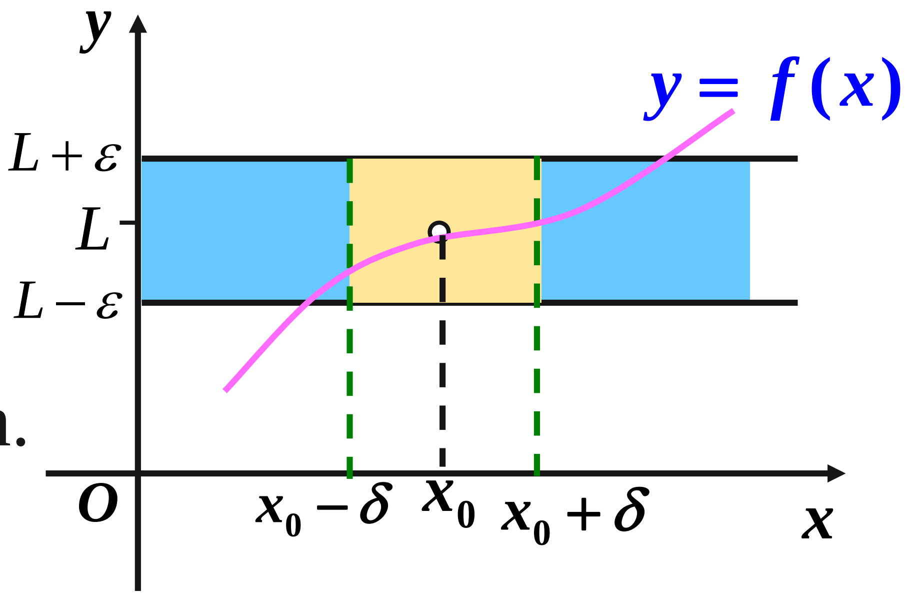
Geometrical meaning of $\lim_{x \to x_0} f(x) = L$

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ when } 0 < |x - x_0| < \delta, \text{ we have } |f(x) - L| < \varepsilon$$

x_0 satisfying $0 < |x - x_0| < \delta$,

$L - \varepsilon < y = f(x) < L + \varepsilon$

The function values of x lie in the domain.



Definition left-hand and right-hand limits

Left-hand limit $\forall \varepsilon > 0,\ \exists \delta > 0,$ such that when $x_0 - \delta < x < x_0$

we have $|f(x) - L| < \varepsilon.$

write it as $\lim\limits_{x \to x_0^-} f(x) = L$ or $f(x_0^-) = L.$

Right-hand limit $\forall \varepsilon > 0, \exists \delta > 0,$ such that when $x_0 < x < x_0 + \delta$

we have $|f(x) - L| < \varepsilon.$

write it as $\lim\limits_{x \to x_0^+} f(x) = L$ or $f(x_0^+) = L.$

Rigorous Study of Limits

$$\{x \mid 0 < |x - x_0| < \delta\}$$

$$= \{x \mid 0 < x - x_0 < \delta\} \cup \{x \mid -\delta < x - x_0 < 0\}$$

$\lim_{x \to x_0} f(x) = L$ $\Longleftrightarrow$ (1) left-hand limit and right-hand limit exist and

(2) the two limits are equal, that is

$$\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x)$$

Example 1

Prove $\lim_{x \to x_0} C = C$, (C is constant).

$\forall \varepsilon > 0$, for any $\delta > 0$, when $0 < |x - x_0| < \delta$,

$|f(x) - L| = |C - C| = 0 < \varepsilon$,

$\therefore \lim_{x \to x_0} C = C.$

Example 2

Prove $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$

$f(x)$ is not defined at $x = 1$

$\because |f(x) - L| = \left|\frac{x^2 - 1}{x - 1} - 2\right| = |x - 1| \quad \forall \varepsilon > 0,$

In order to $|f(x) - L| < \varepsilon,$ need to take $\delta = \varepsilon,$

when $0 < |x - 1| < \delta,$ we have $\left|\frac{x^2 - 1}{x - 1} - 2\right| < \delta = \varepsilon,$

$\therefore \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$

Example 3

Prove $f(x) = \begin{cases} \sqrt{x} & x < 1 \\ \sin x & x \geq 1 \end{cases}$ has no limit when $x \to 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{x} = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sin x = \sin 1$$

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

So, $f(x)$ has no limit when $x \to 1$.

Example 4

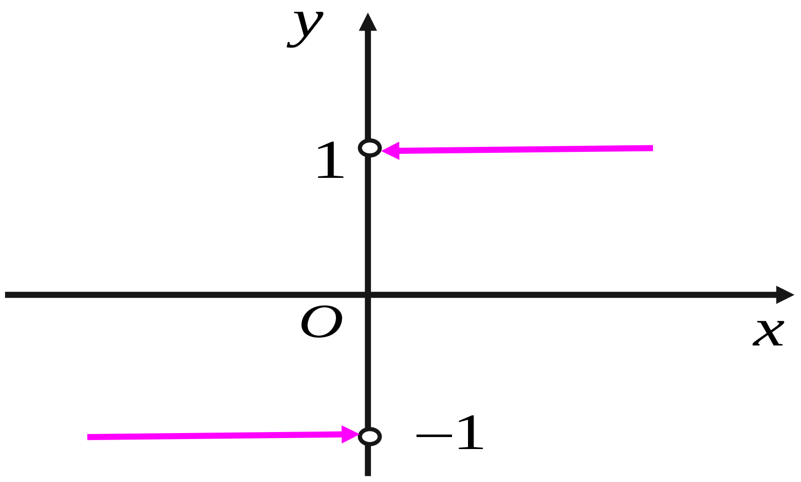
Determine $\lim_{x\to 0}\frac{|x|}{x}$ exists or not.

$$\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$$

$$\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}1=1$$

$$\therefore \lim_{x\to 0^-}\frac{|x|}{x}\neq\lim_{x\to 0^+}\frac{|x|}{x},$$

So the limit does not exist.



Summary

Precise Meaning of Limit $(\varepsilon - \delta)$

If $\forall \varepsilon > 0$, $\exists \delta > 0$, such that when $0 < |x - x_0| < \delta$, we have

$$|f(x) - L| < \varepsilon$$

Questions and Answers

Q1: Prove $\lim_{x \to x_0} x = x_0$.

$\because |f(x) - L| = |x - x_0|$, $\forall \varepsilon > 0$, take $\delta = \varepsilon$,

when $0 < |x - x_0| < \delta = \varepsilon$,

$|f(x) - L| = |x - x_0| < \varepsilon$,

$\therefore \lim_{x \to x_0} x = x_0$.

Questions and Answers

Q2: Prove : when $x_0 > 0, \lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

$|x - x_0| \le x_0$ makes that $x \ge 0$

$$\because |f(x) - L| = |\sqrt{x} - \sqrt{x_0}| = \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| \le \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$$

$\forall \varepsilon > 0,$ in order to $|f(x) - L| < \varepsilon$

Only need to $|x - x_0| < \sqrt{x_0}\,\varepsilon$ and $x \ge 0$

Take $\delta = \min\{x_0, \sqrt{x_0}\,\varepsilon\}$ when $0 < |x - x_0| < \delta$,

we have $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$, $\therefore \lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

Questions and Answers

Q3: Prove $\lim_{x \to 4}(3x-7)=5.$

Analysis Prove that $\lim_{x \to 4}(3x-7)=5.$

$$0<|x-4|<\delta \Rightarrow |(3x-7)-5|<\varepsilon.$$

$$|(3x-7)-5|<\varepsilon \Leftrightarrow |3x-12|<\varepsilon$$

$$\Leftrightarrow 3|x-4|<\varepsilon$$

$$\Leftrightarrow |x-4|<\frac{\varepsilon}{3}.$$

How to choose δ: $\delta=\frac{\varepsilon}{3}.$

Questions and Answers

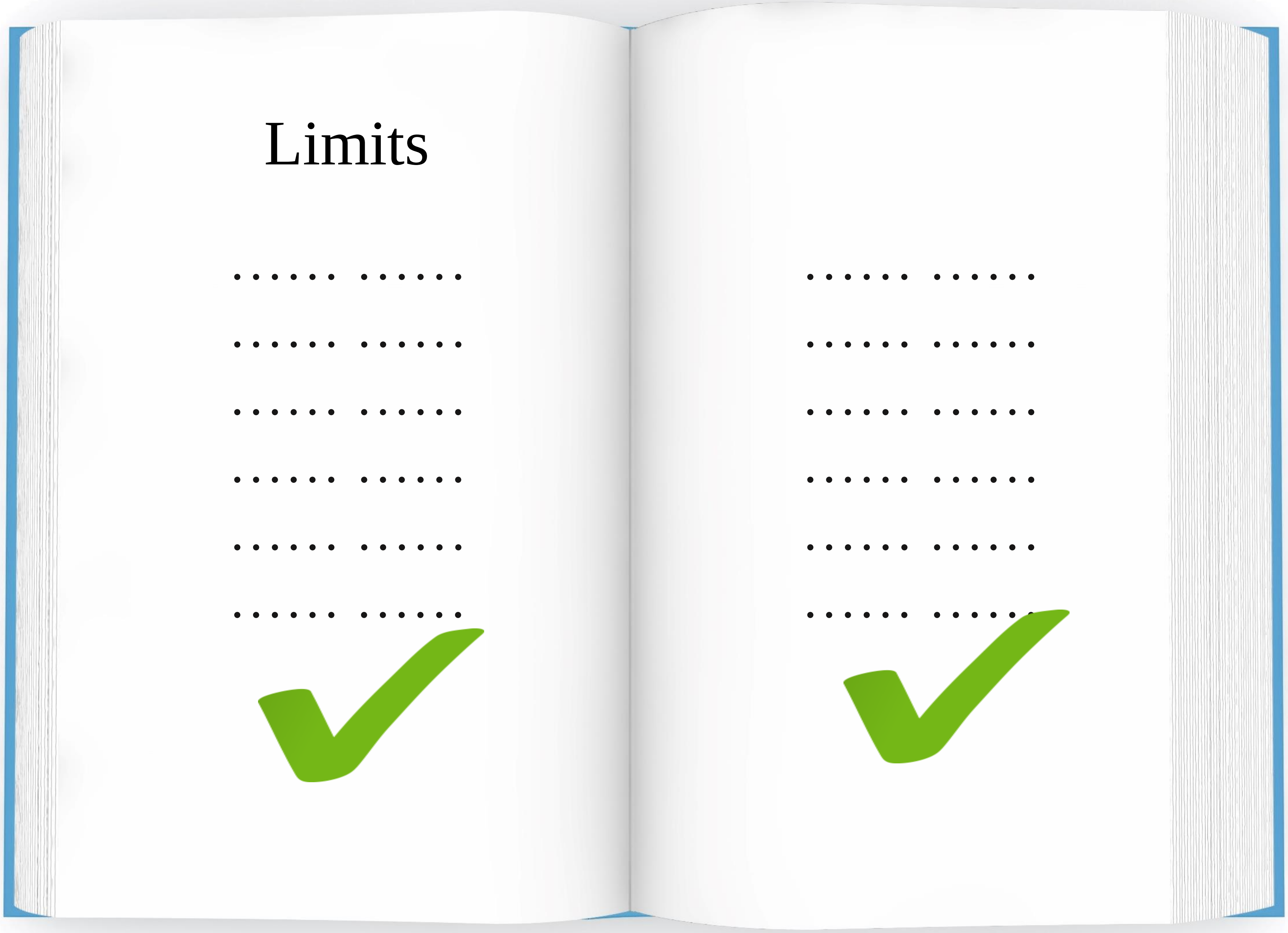
Q3: Prove $\lim_{x\to 4}(3x-7)=5.$

Proof: ① $\forall \varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$.

② $0<|x-4|<\delta$ implies that

③ $|(3x-7)-5| = |3x-12| = 3|x-4| < 3\cdot\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon.$

So we have $|(3x-7)-5| < \varepsilon.$


Limits